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A COMPARISON OF ALTERNATIVE OPTIMAL MODELS OF ADVERTISING EXPENDITURES:STOCK ADJUSTMENT vs. CONTROL THEORETIC APPROACHES APPLIED TO JAPANESE PHARMACEUTICAL COMPANIES

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I. Introduction

Advertising expenditures may well be regarded as a form of investment. Using this concept, Nerlove and Arrow [17] examined an optimal advertising policy for the firm which maximizes present valued cash flow. More recently Gould [9] extended the Nerlove-Arrow model by introducing an adjustment cost function. These studies are theoretical analyses leaving out possible empirical applications.(1) In this paper the Nerlove-Arrow model is applied by use of the usual stock adjustment formula to empirical data. In addition, since the stock adjustment model is made in an ad-hoc fashion, a suboptimization model is presented as an attempt to derive an estimable equation directly from optimization behavior. This model is derived from control theoretic suboptimization procedures incorporating an adjustment cost function. Empirical results from both models are compared. Semi-annual data of eight Japanese pharmaceutical companies from 1963 to 1970 are used for this study.

It seems to have become the predominant practice in econometric studies of dynamic relationships to apply some form of distributed lags to the stock adjustment model

$$(1\text{-}1) \qquad y_t = \lambda(L)[x_t^* - x_{t-1}]$$

where y_t and x_t are respectively dependent and explanatory variables, and x_t^* is the desired level of x_t. $\lambda(L)$ is the distributed lag operator function. In equation (1-1) sometimes x_{t-1}^* is used in place of x_{t-1}. As an explicit form of the distributed lag operator function, $\lambda(L)$, various functional forms and estimation procedures have been suggested [1,6,11,22, 26]. Of late

(1) There are many empirical studies which analyse the cumulative effects of advertising expenditures on sales using some form of distributed lags [2,18,20, 23]. This paper deals with the determination of advertising expenditures.

formulations such as equation (1-1) above have been criticized on the grounds that the distributed lag function is chosen in an ad-hoc fashion without a theoretical justification for the particular lag structures.(2) To make equation (1-1) estimable, one derives a relationship between the desired variable, x_t^*, and observable variables by solving some optimization problems. The derived relationship, then, is taken to be the long-run equilibrium position which the economic system tries to approach by the adjustment equation (1-1). The distributed lag pattern of adjustment, $\lambda(L)$, is superimposed on the system independent of the optimization problem from which x_t^* is derived, and thus it is often hard to explain an estimated lag structure.

One of the ways to overcome the deficiencies of this stock adjustment approach to the distributed lag model is to construct a model incorporating adjustment costs into a dynamic optimization problem which is solved by techniques of optimal control theory. Since Eisner and Strotz [7] introduced the costs of adjustment to derive an investment function, there have been a number of contributions to the theory of investment incorporating the costs of adjustment [4, 10,15,25]. The earlier mentioned paper by Gould [9] is an adaptation of cost of adjustment approach to advertising outlay. All these optimal models treat an infinite time horizon and some of them e.g. Lucas [15], Treadway [25], and Gould [9] are concerned with long-run equilibrium values hence leaving little room for short-run consideration. For systems whose dynamics are deterministic and completely known, it will be reasonable to optimize the objective function over a long period of time. In many practical situations, however, the dynamics of the system may be complex and may undergo unforeseen changes. Furthermore, the mathematical description of the dynamics of the system may be unknown. Then, we may often find it convenient to use a model of assumed mathematical form which may be less complicated than the actual form.(3) For this purpose the time

(2) One of the most recent criticisms on this point was made by Nerlove in his Henry Schultz Memorial lecture to the Second World Congress of the Econometric Society, Cambridge, England, September, 1970 [19].

(3) There are two types of uncertainty which an economic agent faces. One is that precise mathematical forms of behavioral functions and dynamic systems are not known and thus he has to use a model of assumed mathematical form. The second type of uncertainty is that which arises from stochastic systems and variables. These two types of uncertainty can of course co-exist and if one deals with a stochastic system, then stochastic optimization techniques should be used. In this paper we deal with the first type of uncertainty

horizon will be broken into subintervals and time will be treated as a running variable rather than as a fixed variable. One may, then, use Bellman's technique of invariant imbedding equations [3] to handle what is known as a two-point boundary value problem (TPBVP).

In this paper a stock adjustment model of advertising expenditures is compared with a suboptimization model using semi-annual data of eight Japanese pharmaceutical companies. In Section II the stock adjustment model is formulated and empirical results are presented. In Section III the suboptimization model is derived using an invariant imbedding equation applied to the discrete optimization procedure, and empirical results derived from this model are presented. Section IV compares the two approaches.

II. The Stock Adjustment Model of Advertising Expenditures

(i) The formulation of the stock adjustment model: In the Nerlove-Arrow model the demand function of the firm at time t, q(t), is given by

$$(2\text{-}1) \quad q(t) = q[p(t), A(t)]$$

where p(t) and A(t) are respectively the price of q(t) and the net stock of goodwill at time t. The stock of goodwill is a measure of the effects of current and past advertising expenditures, and it is assumed to depreciate at a constant rate, δ, so that

$$(2\text{-}2) \quad a(t) = \dot{A}(t) + \delta A(t)$$

where a(t) and $\dot{A}(t)$ (= $\frac{dA(t)}{dt}$) are respectively gross and net advertising expenditures at time t. Given the cost function C[q(t)], the cash flow at time t, $\Pi(t)$, will be given by

$$(2\text{-}3) \quad \Pi(t) = p(t)q(t) - C[q(t)] - p_a(t)a(t)$$

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which the economic agent faces, partly because of the mathematical tractability of the problem and partly because Japanese pharmaceutical firms treat economic variables and systems as deterministic rather than stochastic.

where $p_a(t)$ is the price level of advertising at time t.(4) Then the goal of the firm is to maximize the present valued cash flow

$$(2\text{-}4) \qquad \int_0^\infty e^{-\rho t}\Pi(t)\, dt$$

subject to the constraint (2-2), where ρ is the discount rate. Then by the method of calculus of variation given in [7: p.46] and after a few steps of manipulation we obtain

$$(2\text{-}5) \qquad A(t) = \frac{\xi}{\eta(\rho + \delta - \dot{p}_a/p_a)} \quad \frac{p(t)q(t)}{p_a(t)}$$

where $\xi = \frac{\partial q(t)}{\partial A(t)} \frac{A(t)}{q(t)}$, and $\eta = - \frac{\partial q(t)}{\partial p(t)} \frac{p(t)}{q(t)}$ are the elasticities of demand with respect to the stock of goodwill and price, respectively, and $\dot{p}_a = dp_a/dt$.

If one believes that in actuality the optimal condition (2-5) were only to be attained through an adjustment of the type given by Jorgenson [12], then one may proceed as follows: equation (2-2) indicates that current advertising expenditures, a(t), consist of net goodwill investment, $\dot{A}$, and replacement, δA. Now suppose the net goodwill investment in a discrete time period, NA_t, is given by

$$(2\text{-}6) \qquad NA_t = \lambda(L)(A_t^* - A_{t-1}^*)$$

where $\lambda(L)$ is the distributed lag function and A_t^* is the desired goodwill at time t. Then the current advertising expenditures, a_t, will be given by

$$(2\text{-}7) \qquad a_t = NA_t + \delta A_{t-1} = \lambda(L)(A_t^* - A_{t-1}^*) + \delta A_{t-1}.$$

If we treat (2-5) as representing the desired net stock of goodwill, $A_t^* = k_1 \frac{p_t q_t}{p_{a,t}}$, then (2-7) now becomes

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(4) Nerlove and Arrow assume the price of advertising to be \$1. Their theoretical model does not lose its generality by this assumption, but in this empirical model we treat the price of advertising as changing over time, using the actual data of advertising price change as discussed in Appendix C.

$$(2\text{-}8) \quad a_t = k_1\lambda(L)(x_t - x_{t-1}) + \delta A_{t-1}$$

where $k_1 = \dfrac{\xi}{\eta(\delta + \rho - \dot{p}_a/p_a)}$, and $x_t = \dfrac{p_t q_t}{p_{a,t}}$. Thus we have obtained the stock adjustment model to determine current advertising expenditures, a_t, as a distributed lag function of past changes in the desired stock of goodwill, $A_t^* - A_{t-1}^*$, and the stock of goodwill in the previous period, A_{t-1}.

What does equation (2-8) imply in terms of actual behavior of the firm ? The equation consists of two parts: net goodwill investment and replacement investment. The replacement portion, δA_{t-1}, is replenished without any delay, whereas as given in equation (2-6) the firm adjusts the net goodwill investment according to discrepancies in the desired stock of goodwill which is expressed as proportional to net sales deflated by the price of advertising, $p_t q_t/p_{a,t}$, an observable variable. The discrepancies of the desired stock of goodwill are assumed to be distributed over time due to lags incurred with appropriations of funds, contracts, and/or delivery of orders [12, p.49]. The lag pattern, $\lambda(L)$, is determined by empirical data. It is important to note here that the lag pattern, $\lambda(L)$, is not given as a consequence of the optimization problem of the firm in equations (2-1)- (2-4) but rather it is superimposed on the firm after this optimization problem is solved as something which reoccurs regularly but unexpectedly and which is beyond the control of the firm.

To estimate equation (2-8) we have to decide on how to measure the stock of goodwill, A_t. We also have to choose the distributed lag function, $\lambda(L)$. Now given the identity

$$A_t = NA_t + \delta A_{t-1}$$

and equation (2-7) we will obtain

$$(2\text{-}9) \quad A_t = a_t + (1-\delta)A_{t-1} = \sum_{k=0}^{\ell}(1-\delta)^k a_{t-k} + (1-\delta)^{\ell} A_{t-\ell}.$$

If δ is sufficiently large or if ℓ is large, we may treat the second term as a constant. Substituting (2-9) into (2-8) we obtain

$$\text{(2-10)} \qquad a_t = k_1\lambda(L)(x_t - x_{t-1}) + \delta(\sum_{k=0}^{\ell}(1-\delta)^k a_{t-k-1}) + \gamma + \varepsilon_t.$$

It is better to have a larger integer value for ℓ on the summation sign of the second term above, but this is constrained by the availability of data. As for the distributed lag function $\lambda(L)$ we use the gamma distributed lags proposed in [26] since they are flexible in the sense that they can include geometrically declining lags (such as Koyck lags) as a special case. Thus we represent $\lambda(L)$ by

$$\text{(2-11)} \qquad \lambda(L) = \{\frac{1}{z}\sum_{k=1}^{n} k^{s-1}\, e^{-k}\, L^{k-1}\}$$

where $z = \Sigma k^{s-1}\, e^{-k}$ and L is the distributed lag operator. To estimate equation (2-10) with $\lambda(L)$ given by equation (2-11) we use the nonlinear least squares method [16]. The use of a nonlinear estimation procedure allows us to estimate the gamma distributed lag parameter s and the rate of depreciation δ. The latter enters in the equation as the $(\ell+1)$th order polynomial.

(ii) The estimation of the stock adjustment model: We used semi-annual data of eight Japanese pharmaceutical companies from 1963 to 1970.(5) The variables used for the estimation of equation (2-10) are

a_t = advertising expenditures at time t deflated by the price index of advertising, $p_{a,t}$, millions of 1965 yen

x_t = net sales at time t deflated by the price of advertising, $p_{a,t}$, millions of 1965 yen.

For the price index of advertising, $p_{a,t}$, we used a price index constructed

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(5) The sample period does not include the initial 7 lags required for the estimation of the gamma lags and stock of goodwill.

Approximately 60 percent of the total industry output is produced by eleven companies. In ethical (prescription) drugs the combined market share of these eleven companies reaches 80 percent. Of these eleven companies, Chugai Pharmaceutical Company was eliminated from the study since it suffered a major sales set back from (continued on next page)

as the weighted sum of price indexes relevant to various advertising categories. (See Appendix C). We set the value of ℓ in the summation sign of the stock of goodwill in equation (2-10) and that of n in the gamma distributed lags in equation (2-10) uniformly at 7 which is the maximum lag number available from our data. The estimated results are tabulated in Table 1 below. $\bar{R}^2$ and DW

Table 1 Estimated Results of the Advertising Expenditures Equation (2-10) by Companies

Company	k_1	s	δ	Constant term	$\bar{R}^2$	DW
Takeda	.4690 (.1289)	4.5795 (.5637)	.1335 (.0373)	2875.31 (1052.19)	.73	2.69
Sankyo	.5393 (.2002)	4.2747 (.8435)	.2368 (.1718)	257.00 (717.36)	.59	2.41
Tanabe	1.0875 (.5638)	2.8392 (.9024)	.3573 (.4104)	-799.80 (165.67)	.56	.71
Fujisawa	1.3183 (.3548)	2.7969 (.5259)	.1525 (.0528)	322.42 (358.99)	.79	2.31
Eisai	1.1601 (.2959)	3.5840 (.4724)	.2384 (.0591)	120.51 (93.85)	.99	3.15
Yamanouchi	.4329 (.2298)	-.7514 (5.0758)	.3954 (.1633)	-18.08 (110.37)	.90	1.79
Banyu	.5520 (.2871)	2.9100 (.7702)	.3278 (.1149)	14.02 (70.50)	.98	2.10
Dainippon	.7027 (.1424)	1.9539 (.4644)	.3192 (.0997)	22.75 (68.44)	.98	2.73

The figures in parentheses just below the estimated coefficients are their estimated standard errors.

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late 1964 to early 1969 when the main product of the company, "Glonsan," which had been estimated to account for more than 40 percent of total sales of the company was branded "useless or even halmful" by the consumer protection movement. It took the company four years to restore net sales to the early 1964 level. The preliminary estimates on two companies, Shionogi and Daiichi, gave negative values for δ with less than unity t-ratios, and the results were omitted from Table 1. The sources of data are described in Appendix C.

denote respectively the coefficient of determination adjusted for degrees of freedom and the Durbin-Watson test statistic.(6)

Table 2 compares the distributed lag structures of the reaction of the sales increment, $(x_t - x_{t-1})$ to advertising expenditures, a_t. From Table 2

Table 2 Distributed Lag Structures of Advertising Expenditures by Companies: $\{\frac{1}{z} k^{s-1} e^{-k}\}$

Company / Time lag	Takeda	Sankyo	Tanabe	Fujisawa	Eisai	Yamanouchi	Banyu	Dainippon
	s= 4.5795	s= 4.2747	s= 2.8392	s= 2.7969	s= 3.5840	s= -.7514	s= 2.9100	s= 1.9539
0	.0283	.0429	.2139	.2220	.1007	.8811	.2008	.4107
1	.1244	.1529	.2816	.2838	.2221	.0963	.2775	.2927
2	.1954	.2122	.2184	.2164	.2329	.0174	.2215	.1585
3	.2013	.2002	.1364	.1335	.1802	.0039	.1411	.0767
4	.1646	.1530	.0756	.0733	.1180	.0010	.0795	.0349
5	.1163	.1022	.0389	.0374	.0695	.0003	.0414	.0153
6	.0743	.0623	.0190	.0182	.0381	.0001	.0205	.0065
7	.0441	.0355	.0089	.0085	.0198	.0	.0097	.0027
8	.0247	.0192	.0041	.0039	.0099	.0	.0045	.0011
Mean	4.58	4.27	2.85	2.80	3.57	1.15	2.92	2.09
Variability	9.16	9.54	5.60	5.56	7.10	1.36	5.97	3.83

*The lines under the figures indicate peaks of the lags.

(6) Since both the dependent variables, $x_t - x_{t-1}$, and A_{t-1}, and the dependent variable, a_t, in equation (2-10), are deflated by the price index of advertising, a suprious association between deflated sales and deflated advertising expenditures may be introduced. Hence, we estimated the equation in undeflated form to examine to what extent the results in Table 1 are sensitive to the use of this price index. The results indicate that the estimated values of k_1, s, and δ are quite close to those in Table 1 (variations being less than ten percent) but the constant terms are affected considerably, in some cases showing a 30 percent variation. The coefficient of determination tended to be higher for the estimated equation in undeflated form while the values of the Durbin-Watson test statistics were similar to those in Table 1.

it is clear that the estimated distributed lag structures vary considerably from company to company: at one extreme Takeda attains a peak lag period of 3 with a mean lag of 4.58, while at another extreme, Yamanouchi and Dainippon have peaks in the current period with mean lags of 1.15 and 2.09 respectively. If distributed lag structures and depreciation rates of goodwill vary widely from company to company, as exhibited in Table 2, this calls into question the validity of an aggregative industry study which is based on data obtained by summing individual company data.

What are the reasons why we have a wide range of distributed lag structures ? As discussed earlier the lag function $\lambda(L)$ in equation (2-10) was superimposed on the system independent of the optimization problem described in equations (2-1) to (2-4). Hence, one has no recourse to a theoretical justification for the lag-structures; that is, we cannot specify that a particular lag pattern arises as a result of the optimization behavior process of the firm. Thus one has to turn to other sources for a possible explanation, sources such as unforseen delivery lags or some delays caused in the organizational process of appropriation of funds, letting of contracts, issuing of orders,etc. In this connection one possible explanation is that Takeda, the largest company in terms of sales and employment as exhibited in Table 3 below has the slowest reaction lag with a mean lag of 4.58 since a large organization tends to generate more organizational bureaucracy in processing decisions. However, this is a hypothesis to be tested by careful study of the organization and decision making processes of these various companies. At any rate it is clear that the superimposition of a lag function in an ad-hoc manner leaves the researcher with a heavy task of interpretation of results.(7)

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(7) This result somewhat resembles the results of principal component and/or factor analyses which are more frequently used in biometrics and psychometrics since after extracting factors the researcher has to interpret them. However, in these disciplines the task is more clearly defined than in an interpretation of a distributed lag pattern since the researcher will be dealing with controlled data with the structure of the system known to him.

Table 3 Net Sales, Growth Rates and Total Employment of Top Ten Companies

Company	Net Sales at the end of first half of 1970 (100 million yen)	Growth Rate (1970.I/1965.I) (percentage)	Total Employment (1970.I) (Persons)
Takeda	901	80.2	11,991
Sankyo	286	71.3	6,222
Shionogi	235	49.7	6.292
Tanabe	239	41.1	5,232
Fujisawa	182	122.0	3,586
Eisai	172	164.6	2,315
Yamanouchi	164	137.7	3,431
Banyu	156	178.6	1,650
Daiichi	136	67.9	2,978
Dainippon	101	134.9	2,174

Source: Security and Exchange Reports of the Companies to the Minister of Finance, various issues.

In estimating the gamma distributed lags in equation (2-10) we had to limit the number of summation terms, n, in equation (2-11) to some finite number and in our case we set the number at seven which is the maximum of lags available from our data. By so doing, we are committing a misspecification error which arises by approximating an infinite lag structure by a finite lag structure. As discussed in Dhrymes [5,p.45 and p.52] such misspecification is likely to affect the mean lag and variability of the lag distribution. The mean lag, m(s), and variability, V(s), of the gamma distributed lag are given by

$$(2\text{-}12) \qquad m(s) = \frac{z(s+1)}{z(s)}$$

$$(2\text{-}13) \qquad V(s) = \frac{z(s+2)}{z(s)} + \frac{z(s+1)}{z(s)} - \left[\frac{z(s+1)}{z(s)}\right]^2$$

where $z(s)=\sum_{k=1}^{\infty} k^{s-1}e^{-k}$. From the numerical calculations we find that $m(s) \doteqdot s$ and $V(s) \doteqdot 2s$ when s is larger than three. Consequently the mean lag and variability of the gamma distributed lag depend crucially on the estimated value of s. To examine its sensitivity to the change in the number of summation, n, in equation (2-11), we estimated equation (2-10) for n=5 and n=6 for each company, and found that the estimated values of s did not change more than five percent from those given in Table 1.

The nonlinear estimation of equation (2-10) gives us an estimate of the depreciation rate of goodwill which varies between 13.4 percent and 39.5 percent per half year period. This depreciation rate, as noted before, is estimated with past data on advertising expenditures going back seven periods. It may be at best interpreted as the individual company's subjective evaluation of the rate of depreciation of goodwill rather than the actual depreciation rate which would reflect the rate at which the consumers' memory of a product fades.

The stock adjustment model estimated above seems in general to be able to explain the data judged by such standard measures as coefficient of determination, Durbin-Watson test statistics, the sizes of the estimated standard errors and the signs of estimated coefficients. However, the model does not explain the distributed lag patterns.

III. Suboptimization Model of Advertising Expenditures

(i) Formulation of the suboptimization model: It is often argued that an equation to be estimated, including a particular time (lag) structure, has to be derived from an optimization problem. A stock adjustment model fails to do this. One of the efforts in this direction is to construct a model incorporating adjustment costs into a dynamic optimization problem. In this section we shall formulate the dynamic problem of optimal advertising policy by incorporating adjustment costs into a dynamic optimization problem. We shall use a discrete maximum principle for this purpose.(8) Suppose that the firm plans

(8) In most of the literature the optimal control problems are formulated by using a continuous maximum principle with an infinite time horizon, and just before the estimation of a model derived from the continuous maximum principle, it is changed to a discrete model. This procedure was followed in Section II. To an econometrician who uses discrete time series data it

to maximize the present value of cash flow over the period from 0 to t_f:

$$(3\text{-}1) \qquad V = \sum_{t=0}^{t_f} \frac{\Pi(t)}{(1+\rho)^t}$$

where $\Pi(t)$ is defined as $\Pi(t) = p(t)q(t) - C[q(t)] - S(t)$ and ρ is the discount rate. The model we work with is the same as before except that equation (2-2) now becomes

$$(3\text{-}2) \qquad A(t+1) = a(t) + (1-\delta)A(t)$$

and we introduce an adjustment function, $S(t)$.

As Nerlove and Arrow indicate [17, p.130], it will be more realistic to think that the cost of advertising activities, $S(t)$, tends to increase at an accelerated rate with the quantity and intensity of advertising expenditures, for at high levels of such expenditure some inferior media may have to be used. Furthermore, if the price level of advertising, $p_a(t)$, increases relative to the wage and salary rate of the firm, $w(t)$, the firm may be encouraged to carry out some advertising activities by itself rather than contract them out to advertising agents.(9) Then, the minimum conditions which the function, $S(t)$, should satisfy may be

$$\frac{\partial S(t)}{\partial a(t)} > 0, \quad \frac{\partial S(t)}{\partial p_a(t)} > 0, \quad \frac{\partial S(t)}{\partial w(t)} < 0, \text{ and } \frac{\partial^2 S(t)}{\partial a(t)^2} > 0.$$

The simplest specification satisfying these conditions is a quadratic form in $a(t)$:

$$(3\text{-}3) \qquad S(t) = \alpha_0 a(t)^2 + \alpha_1 \frac{p_a(t)}{w(t)} a(t), \qquad \alpha_0, \alpha_1 > 0$$

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will be better to formulate the optimal problem using a discrete maximum principle. In many cases the firm will plan strategies in a fixed time horizon. Consequently, we formulate the model in this section using a discrete maximum principle with a finite time horizon, $[0, t_f]$.

(9) Among Japanese pharmaceutical companies, given a rise in price level of advertising, the tendency would be to rely more heavily on detail salesmen who are permanent employees of the company to carry out sales promotion.

Now the optimal problem is to maximize equation (3-1) subject to equations (3-2) and (3-3). As shown in Appendix A, we derive the optimum trajectory of the advertising equation and its adjoint equation as

$$(3\text{-}4) \qquad a(t) = \frac{1}{2\alpha_0(1+\rho)}\lambda(t+1) - \frac{\alpha_1}{\alpha_0}\frac{p_a(t)}{w(t)}$$

$$(3\text{-}5) \qquad \lambda(t+1) = -\frac{(1+\rho)}{(1-\delta)\eta}p(t)q_A(t) + \frac{1+\rho}{1-\delta}\lambda(t)$$

where $q_A(t)=\frac{\partial q(t)}{\partial A(t)}$, and $\eta = -\frac{\partial q(t)}{\partial p(t)}\frac{p(t)}{q(t)}$, i.e. the price elasticity of demand.

If we solve the first order difference equation (3-5) for $\lambda(t+1)$, then from equation (3-4) we can determine $a(t)$. However, there are two main difficulties one often faces in solving the difference equation, i.e. the split boundary condition or the two point boundary value problem (TPBVP) and the unknown form of the demand function $q(t)$ $q[p(t),A(t)]$. The problem of TPBVP arises, since, for example, the firm will know the quantity of goodwill at the initial time, $t=0$, i.e. $A(0)=A_0$, and furthermore the firm may expect the transversality condition (A-7) in Appendix A which states that the firm expects the price of advertising at the terminal period t_f to be $p_a^e(t_f)$, but the initial condition of $\lambda(0)$ is not known.

To get around these two difficulties we use the invariant imbedding procedure [3,21] in which the missing initial (or terminal) condition is obtained in a direct manner, and thus the original problem is effectively reduced to an initial-value problem.(10) To derive the invariant imbedding equation let us rewrite equations (3-2) and (3-5) as follows:

$$(3\text{-}6) \qquad A(t+1) = a(t) + (1-\delta)A(t) = f[A(t),\lambda(t+1),t]$$

$$(3\text{-}7) \qquad \lambda(t+1) = -\frac{1+\rho}{1-\delta}p(t)q_A(t) + \frac{1+\rho}{1-\delta}\lambda(t) = g[A(t),\lambda(t),t]$$

(10) As we see from the succeeding discussion the invariant imbedding technique crucially depends on an assumption that the unknown forms of functions in the system can be approximated locally by linear functions.

The boundary conditions are given by

$$A(0) = A_o, \text{ and } \lambda(t_f) = p_a^e(t_f).$$

The process starts at $t_o = 0$, and ends at t_f. We let

$$A_o = K, \text{ and } \lambda(0) = r(K,0) \tag{3-8}$$

and thus we have imbedded the initial condition $A(0)=A_o$ in a more general class of initial conditions, K, and we are letting $r(K,t)$ denote the missing initial condition of $\lambda(0)$.

Now as usual we make an assumption that $r(K,t)$ is approximated by a linear function of K such that

$$r(K,t) = d(t) - u(t)K. \tag{3-9}$$

Then we obtain the invariant imbedding equation

$$\begin{aligned} d(t+1)- d(t) -[u(t+1)-u(t)]K - u(t+1)[f(K,r,t) -K] \\ = g(K,r,t) - r(K,t). \end{aligned} \tag{3-10}$$

In equation (3-5) we do not have an explicit mathematical function for $q_A(t)$. If the demand function is at least twice differentiable with respect to $A(t)$, we may expand $q_A(t)$ in a Taylor series about $K=A_o$ to obtain

$$q_A(K) = q_A(A_o) + q_{AA}(A_o)(K -A_o) \tag{3-11}$$

where $q_A(A_o) = \left.\frac{\partial q(t)}{\partial A(t)}\right|_{A(t)=A_o}$, and $q_{AA}(A_o) = \left.\frac{\partial^2 q(t)}{\partial A(t)^2}\right|_{A(t)=A_o}$.

As shown in Appendix B, we derive the following Riccati-type difference equation in $d(t)$ and $u(t)$:

$$(3\text{-}12)\qquad d(t+1) - \frac{1+\rho}{1-\delta}\,d(t) - \frac{1}{2\alpha_o(1+\rho)}d(t+1)u(t+1) + \frac{\alpha_1}{2\alpha_o}\,\frac{p_a(t)}{w(t)}\,u(t+1)$$

$$+ \frac{1+\rho}{(1-\delta)\eta}\,p(t)q_A(A_o) - \frac{1+\rho}{(1-\delta)\eta}p(t)q_{AA}(A_o)A_o = 0$$

$$(3\text{-}13)\qquad (1+\delta)u(t+1) - \frac{1+\rho}{1-\delta}\,u(t) + \frac{1}{2\alpha_o(1+\rho)}u(t+1)^2 - \frac{1+\rho}{(1-\delta)\eta}p(t)q_{AA}(A_o)=0.$$

Equations (3-12) and (3-13) can be solved backwards starting from the terminal condition $\lambda(t_f)= p_a^e(t_f)$: from equations (3-8) and (3-9) we see that

$$(3\text{-}14)\qquad \lambda(t) = r(K,t) = d(t) - u(t)K.$$

Then at $t=t_f$ we must have $d(t_f)=p_a^e(t_f)$ and $u(t_f)=0$ to satisfy the terminal condition $\lambda(t_f)=p_a^e(t_f)$. At $t=t_f-1$ we use $u(t_f)=0$ to obtain a new value for $u(t_f-1)$ from equation (3-13):

$$u(t_f-1) = -\frac{1}{\eta}\,p(t_f-1)q_{AA}(A_o)$$

and in turn this value is used to solve for $d(t_f-2)$. Then we solve for $\lambda(t_f-1)$ from (3-14) noting that $K \doteqdot A_o$. Once this is done, then from equation (3-4) we can find a value for $a(t_f-2)$.

To solve for $a(t)$ in the manner explained above we are required to know the values of parameters, α_o, α_1, δ, and ρ as well as the values of $p(t)$, $C'(t)$, and $q_{AA}(t)$. If only some of these parameters, say, α_o and α_1 are unknown, then we may treat them as the "state variables" such that $\alpha_o(t+1)=\alpha_o(t)$, and $\alpha_1(t+1)=\alpha_1(t)$, and by incorporating them into our system [i.e. (3-4) and (3-5)] we may develop "on-line" estimation of states and parameters by applying the least squares criterion as the cost function. However, if the number of unknown parameters to be estimated is large, one may not easily obtain stable estimates of states and parameters. Furthermore, we do not know the mathematical form of the demand function, $q(t)$, and the values of $p(t)$, for all $t \in [0, t_f]$, may not easily predicted. As we see in the succeeding discussion, however, the estimation of states and parameters could be considerably simplified in order to be handled by ordinary regression analysis. This simplification hinges on the assumption that we can treat $q_{AA}=0$ for all $t \varepsilon[0, t_f]$.

If $q_{AA}=0$ for all $t \varepsilon[0, t_f]$, then given the condition that $u(t_f)=0$, it follows from equations (3-12) and (3-13) that $u(t)=0$ for all $t \varepsilon[0, t_f]$ and then equation (3-12) becomes

$$(3\text{-}15) \qquad d(t+1) - \frac{1+\rho}{1-\delta} d(t) + \frac{1+\rho}{(1-\delta)\eta} p(t) q_A(t) = 0$$

or the same as equation (3-5) with $\lambda(t)= d(t)$. Given the terminal condition $\lambda(t_f)= p_a^e(t_f)$ again equation (3-15) can be solved backwards to yield

$$(3\text{-}16) \qquad d(t) = \left[\frac{1-\delta}{1+\rho}\right]^{t_f-t} p_a^e(t_f) + \frac{1}{\eta} \sum_{\tau=t}^{t_f-1} \left(\frac{1-\delta}{1+\rho}\right)^{\tau-t} p(\tau) q_A(\tau).$$

Then substituting equation (3-16) into (3-4)

$$(3\text{-}17) \qquad a(t)= \frac{1}{2\alpha_o(1+\rho)}\left[\frac{1-\delta}{1+\rho}\right]^{t_f-t-1} p_a^e(t_f)$$

$$+ \frac{1}{2\alpha_o(1+\rho)\eta} \sum_{\tau=t+1}^{t_f-1} \left(\frac{1-\delta}{1+\rho}\right)^{\tau-t-1} p(\tau) q_A(\tau) - \frac{\alpha_1}{2\alpha_o} \frac{p_a(t)}{w(t)}$$

or

$$a(t) = \beta_o + \beta_1 \sum_{\tau=t+1}^{t_f-1} \left(\frac{1-\delta}{1+\rho}\right)^{\tau-t-1} p(\tau) q_A(\tau) + \beta_2 \frac{p_a(t)}{w(t)}$$

where $\beta_o = \frac{1}{2\alpha_o(1+\rho)}\left[\frac{1-\delta}{1+\rho}\right]^{t_f-t-1} p_a^e(t_f)$, $\beta_1= \frac{1}{2\alpha_o(1+\rho)\eta}$, and $\beta_2= - \frac{\alpha_1}{2\alpha_o}$.

Equation (3-18) may be estimated by an appropriate regression method, if one knows the time of the planning horizon, t_f, and the marginal efficiency of advertising $q_A(t)$. Since in many practical cases entrepreneurs tend to rely on an average efficiency as the measure of a marginal efficiency performance, we may represent

$$\text{(3-19)} \qquad q_A(t) = \gamma \frac{q(t)}{A(t)}.$$

Substituting equation (3-19) into (3-18) we obtain

$$\text{(3-20)} \qquad a(t) = \beta_0 + \beta_1' \sum_{\tau=t+1}^{t_f-1} \left(\frac{1-\delta}{1+\rho}\right)^{\tau-t-1} \frac{p(\tau)q(\tau)}{A(\tau)} + \beta_2 \frac{p_a(t)}{w(t)}$$

where $\beta_1' = \gamma\beta_1$, and we expect that $\beta_1' > 0$ and $\beta_2 < 0$.

Equation (3-20) implies a Koyck type distributed lag function of finite order with lags extending into the future time periods up to t_f-1.(11) What does equation (3-20) mean in terms of what the firm actually might do ? The firm will examine the discounted value of advertising efficiency over the remaining planning period, $[t+1, t_f-1]$, and weigh it against the ratio of the current price of advertising to the current wage rate in order to arrive at an advertising investment decision. Consequently equation (3-20) says that the advertising expenditures of the firm are decided on the cost-benefit basis: cost is the market price of advertising, while the average efficiency of advertising serves as the benefit criterion. The stock adjustment model of equation (2-8) does not incorporate this market efficiency criterion: there,

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(11) In fact from equation (3-20) we obtain the following Koyck type transformation:

$$a(t) - \frac{1-\delta}{1+\rho} a(t+1) = \left(\frac{\rho+\delta}{1+\rho}\right)\beta_0 + \beta_1' \frac{p(t+1)q(t+1)}{A(t+1)}$$
$$+\beta_2\left[\frac{p_a(t)}{w(t)} - \frac{1-\delta}{1+\rho} \frac{p_a(t+1)}{w(t+1)}\right] + u_t - \frac{1-\delta}{1+\rho} u_{t+1}$$

where u_t is the disturbance term. One may estimate the above equation instead of equation (3-20) by devising a gneralized least squares estimation procedure to take care of the autocorrelation generated by the transformation. This procedure is discussed in [27], and we tried it for our data here. However, the results were much poorer than those given in Table 4 judged by such conventional criteria as the goodness of fit, t-ratios and expected sign of estimated coefficients.

the advertising expenditures are determined by more physical factors such as the magnitudes of discrepancies of the desired stock of goodwill and the speed of adjustment expressed by the distributed lag function, $\lambda(L)$.

(ii) The estimation of the suboptimization model: Equation (3-20) was estimated by the ordinary least squares method for each of the eight companies by setting $t_f = t+4$, indicating the assumption that the companies set two years as a sales planning period. As estimates of δ we used those in Table 1 and the value of ρ was set to be .09. A(t) is estimated by

$$A(t) = \sum_{k=1}^{8} (1-\delta)^k a(t-k) .$$ (12)

Only three out of the eight companies gave reasonable results with the right expected signs: $\beta_1 > 0$ and $\beta_2 < 0$, while the remaining five companies had wrong signs for these parameter estimates. Table 4 presents the estimated results for the three companies which had the right signs.

Table 4 The Estimates of the Suboptimization Model

Company	Constant term	β_1	β_2	A priori given $(1-\delta)/(1+\rho)$	$\bar{R}^2$	DW
Fujisawa	1917.69 (2006.83)	406.6196 (311.5354)	-486.0017 (1333.6921)	.7775	.65	1.43
Eisai	72.82 (1658.23)	978.9731 (252.1744)	-468.5730 (1024.7966)	.6987	.96	1.33
Banyu	-1619.93 (1602.44)	597.1663 (151.1973)	-312.1506 (846.7153)	.6167	.86	.92

The figures in parentheses just below the estimated coefficients are their estimated standard errors.

The estimates of β_2 for these three companies were negative but standard error estimates were large.

(12) The discounted value of 9 percent was chosen on the basis that net profits after taxes averaged between 8 and 10 percent of net sales of these eight pharmaceutical companies over the sample period. As an alternative measure of A(t) the sample average of a(t) over the past seven periods was used and the results were similar to those obtained above.

IV. Comparison of Stock Adjustment with Suboptimization Models

In the previous sections we have derived and estimated stock adjustment and suboptimization models of advertising expenditures. From an empirical point of view it may be argued that the validity of an econometric model should be judged by how well it will explain observed data. The stock adjustment model explains advertising expenditures of the eight companies reasonably well, whereas only three companies comply with the suboptimization model judged by the right expected signs of the estimated coefficients. Among the three companies can one choose one model over the other based on the power of explaning data ?

A comparison of alternative econometric models is often made on the basis of goodness of fit and the Durbin-Watson test statistic [12]. Since our models involve lagged endogenous variables and since the stock-adjustment model is estimated by the nonlinear least squares method, the Durbin-Watson test statistic should be cautiously applied and furthermore goodness of fit, as measured by the coefficient of determination or by the standard error of regression, does not permit strict significant tests to discriminate, for example, R^2 between .99 and .96. Goodness of fit criterion, then, has to be taken as a loose measure. Because we do not have more clear-cut measures, let us use the conventional criteria of goodness of fit and the Durbin-Watson test statistic.

Table 5 below presents three measures for each of the three companies whose advertising expenditures seem to be explained either by the stock adjustment or suboptimization models. These three measures are the coefficient of determination corrected for degrees of freedom, $\bar{R}^2$, the estimated standard error of the fitted residuals corrected for degrees of freedom, s_a, and the Durbin-Watson test statistic, DW.

Table 5 Goodness of Fit Criteria

Company	Stock Adjustment Model			Suboptimization Model		
	$\bar{R}^2$	s_a	DW	$\bar{R}^2$	s_a	DW
Fujisawa	.79	172.45	2.31	.65	221.99	1.43
Eisai	.99	73.02	3.15	.96	133.27	1.33
Banyu	.98	64.02	2.10	.86	164.87	.92

Both the coefficients of determination corrected for degrees of freedom and the estimated standard errors of the fitted residuals indicate that the stock adjustment model explains the advertising expenditures of these three companies better than the suboptimization model. The Durbin-Watson test statistics do not present a clear-cut picture except that in the case of Banyu its value changes from 2.10 in the stock adjustment model to .92 in the suboptimization model. This may be due to the fact that the suboptimization model involves a more restricted form of distributed lag and this gives rise to more autocorrelation of the observed residuals.

Another important criterion for the validity of an econometric model is its power in predicting the future, and thus we should compare the predictive performance of the stock adjustment and suboptimization models. If a model involves lagged endogenous variables, their computed values should be used in place of actual values as the former become available. In this vein we shall make some comparisons of the predictive performance of these two models.

To use the suboptimization model (3-20) for prediction we face a difficulty since we need the future values of $p(\tau)q(\tau)/A(\tau)$ for $\tau\varepsilon[t+1,\ t_f-1]$. Hence let us postulate that the firm expects that for the planning period the future value of average efficiency of advertising is given by the current value:

$$(4\text{-}1) \qquad p(\tau)q(\tau)/A(\tau) = p(t)q(t)/A(t), \quad \text{for} \quad \tau\varepsilon[t+1,\ t_f-1].$$

Equation (3-20), then, becomes

$$(4\text{-}2) \qquad a(t) = \beta_0 + \beta_1'' \frac{p(t)q(t)}{A(t)} + \beta_2 \frac{p_a(t)}{w(t)}$$

where $\beta_1'' = \beta_1'[\sum_{\tau=t+1}^{t_f-1} (\frac{1-\delta}{1+\rho})^{\tau-t-1}]$. The estimated results for the three companies are presented in Table 6.

Table 6 The Estimates of Parameters in Equation (4-2)

Company	Constant term	β_1''	β_2	$\bar{R}^2$	DW
Fujisawa	794.99 (1336.62)	1702.7471 (598.8218)	-13.6319 (824.9221)	.76	1.48
Eisai	504.86 (1053.43)	2282.4400 (398.5664)	-908.5815 (628.4026)	.97	1.54
Banyu	-859.98 (1421.83)	1213.4792 (308.3677)	-1023.4177 (696.2212)	.86	1.24

The figures in parentheses just below the estimated coefficients are their estimated standard errors.

Using the estimated equation (4-2) we made a sample period simulation for each of the three companies and compared the results with a sample period simulation obtained from the estimated stock adjustment equation (2-8) by Theil inequality coefficents [24].(13) The results are given in Table 7. In all three cases, the stock adjustment model yields better results than the suboptimization model, although one should not put too much weight on differences of magnitude from .011 to .068, since there is no statistical basis for discriminating them.

Table 7 The Theil Inequality Coefficients

Company	Stock adjustment equation (2-8)	Suboptimization equation (4-2)
Fujisawa	.029	.032
Eisai	.011	.019
Banyu	.017	.068

Based on the conventional criteria of goodness of fit and of predictive performance one can argue that the stock adjustment model explains the advertising expenditure better than the suboptimization model. Consequently, one

(13) As discussed in [14], all measures of predictive performance tend to suffer from biases, and no statistical tests based on small samples are available. Hence, the Theil inequality coefficients above should be taken as a rough indication.

may argue that despite the lack of theoretical justification for lag adjustment structures, the stock adjustment model can be used for prediction if the estimated lag structure is stable with respect to variations in data and the stochastic assumptions underlying the model. However, this still leaves the uneasiness that the model is without a theory. After all, it has been observed in many empirical studies that many types of gradual adjustment mechanisms will yield a behavioral history that some flexible distributed lag schemes "explain" quite well. All of the existing lag estimation procedures are designed to choose a lag structure such that it will best explain the past data. On the other hand the suboptimization model (3-20) has a restricted form of Koyck-type time pattern with the lag parameter $(1-\delta)/(1+\rho)$ given a priori rather than being determined by the data.

The suboptimization model was given here as an attempt to derive an equation to be estimated directly from an optimization problem of the firm and thus as an effort to present a measurement with theory. In order to derive the equation, however, it was necessary to impose various assumptions, and they were made based on knowledge of how Japanese pharmaceutical firms actually behave. For example, the adjustment cost function was made to reflect the trade-off between the price of advertising and the wage rate and the firm was assumed to treat $q_{AA}(t)=0$ during the planning period and the boundary conditions were $A(0)=A_0$, and $\lambda(t_f)= p_a^e(t_f)$. If one treats $q_{AA}(t)$ as nonzero, then it is clear from equations (3-12) and (3-13) that one cannot derive any analytical solution. This is one example of how quickly a solution of an optimization model can run into complications.

Appendix A Derivation of Equations (3-4) and (3-5) Using the Discrete Maximum Principle

The problem is to maximize

$$V = \sum_{t=0}^{t_f} \frac{\Pi(t)}{(1+\rho)^t} \tag{A-1}$$

given the discrete system

$$A(t+1) = a(t) + (1-\delta)A(t) \tag{A-2}$$

where $\Pi(t) = p(t)q(t) - C[q(t)] - S(t)$. In this model the control variables are advertising expenditures, $a(t)$ and price, $p(t)$. Now the Hamiltonian is given by

$$H(t) = \frac{1}{(1+\rho)^t}\,\Pi(t) + \frac{1}{(1+\rho)^{t+1}}\lambda(t+1)A(t+1). \tag{A-3}$$

Using the Pontryagin maximum principles for the discrete system [21, p.129] we find from $\frac{\partial H(t)}{\partial a(t)} = 0$ and $\frac{\partial H(t)}{\partial p(t)} = 0$ that

$$a(t) = \frac{1}{2\alpha_0(1+\rho)}\lambda(t+1) - \frac{\alpha_1}{\alpha_0}\,\frac{p_a(t)}{w(t)} \tag{A-4}$$

$$[p(t) - C'(t)]\,\frac{\partial q(t)}{\partial p(t)} + q(t) = 0. \tag{A-5}$$

From $\frac{1}{(1+\rho)^t}\lambda(t) = \frac{\partial H(t)}{\partial A(t)}$ we obtain

$$\lambda(t+1) = \frac{1+\rho}{1-\delta}[p(t) - C'(t)]q_A(t) + \frac{1+\rho}{1-\delta}\lambda(t) \tag{A-6}$$

where $q_A(t) = \frac{\partial q(t)}{\partial A(t)}$, $C'(t) = \frac{\partial C(t)}{\partial q(t)}$, and $\eta = -\frac{\partial q(t)}{\partial p(t)}\,\frac{p(t)}{q(t)}$.

The transversality condition may be given as

$$\lambda(t_f) = p_a^e(t_f) \tag{A-7}$$

where $p_a^e(t_f)$ may be interpreted as the expected price of advertising at the terminal period t_f. From equation (A-5) we obtain the following optimal price policy:

$$\text{(A-8)} \qquad p(t) = \frac{\eta}{\eta - 1} C'(t).$$

Substituting this into equation (A-6) for C'(t), we obtain

$$\text{(A-9)} \qquad \lambda(t+1) = - \frac{1+\rho}{(1-\delta)\eta} p(t)q_A(t) + \frac{1+\rho}{1-\delta} \lambda(t).$$

Thus equations (A-4) and (A-9) are, respectively, equations (3-4) and (3-5) in the text.

Appendix B Derivation of Equations (3-12) and (3-13)

Substituting equations (3-6) and (3-17) into equation (3-10) we obtain

$$\text{(B-1)} \qquad d(t+1)- d(t) - [u(t+1) -u(t)]K - u(t+1)\{[a(t)- (1-\delta)K] - K\}$$
$$= - \frac{1+\rho}{(1-\delta)\eta} p(t)q_A(K) + \frac{1+\rho}{1-\delta} \lambda(t) - d(t) + u(t)K .$$

Equation (3-8) implies $\lambda(t+1) = r(K, t+1)$, and by equation (3-9) we have

$$\text{(B-2)} \qquad \lambda(t+1) = d(t+1) - u(t+1)K$$

and substituting (B-2) into equation (3-4) we obtain

$$\text{(B-3)} \qquad a(t) = \frac{1}{2\alpha_0(1+\rho)}[d(t+1) - u(t+1)K] - \frac{\alpha_1}{2\alpha_0} \frac{p_a(t)}{w(t)} .$$

We substitute equations (B-2) and (B-3) into (B-1) and equate terms of likewise powers in K (i.e. constant term and K) in equation (B-1) to obtain equations (3-12) and (3-13).

Appendix C Sources of Data

$a(t)$ = from the income statement of each company for an accounting period (6 months), items covering advertising and sales promotion expenditures are summed up. These expenditures are classified under such headlines as "advertising expenditures," "sales promotion expenditures," "entertainment expenditures," and "correspondence and travelling expenses." The definition and title of each item varies from company to company.

$x(t)$ = net sales value reported in the income statement of each company for an accounting period, deflated by the price index of advertising.

$p_a(t)$ = the price index of advertising constructed as the weighted sum of indexes of service charges by categories (source: Economic Statistics Annual, Bank of Japan, various issues). The weights are based on the average advertising expenditures by categories for 30 major commodities of Eisai products in the period 1960-1969. The expenditure categories included TV and newspaper advertising, direct-mail, magazine and medical journal advertising, and expense account entertaining. These categories were arranged to agree with those given in the indexes of service charges. The detailed advertising expenditure accounts were made available to the author through the courtesy of the Eisai Company. Since detailed advertising expenditure accounts of other companies are not available to the author, Eisai's accounts were used as weights to calculate the price index of advertising.

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